Percolation threshold of correlated two-dimensional lattices

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Previous simulations of percolation on correlated square and cubic lattices [Phys. Rev. E **56**, 6586 (1997)] have been extended to all of the common two-dimensional lattices, including triangular, square 1-2, honeycomb, and kagome. Simulations were performed on lattices of up to 1024×1024 sites. The results are independent of lattice size except, possibly, for a weak dependence at large correlation lengths. As in the previous studies, all results can be fit by a Gaussian function of the correlation length w, $p_c = p_c^\infty + (p_c^0 - p_c^\infty) e^{-\alpha w^2}$. However, there is some evidence that this fit is not theoretically significant. For the self-matching triangular and the matching square and square 1-2 lattices, the percolation thresholds satisfy the Sykes-Essam relation $p_c(L) + p_c(L^*) = 1$. [S1063-651X(99)09712-3]

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I. INTRODUCTION

In an earlier paper [1] I reported simulations of percolation on correlated two- and three-dimensional lattices. The correlations followed a model of Crossley, Schwartz, and Banavar [2] that has been used by several authors [3–8] to model random materials. Crossley, Schwartz, and Banavar determine whether a lattice site is open or closed by generating a random function $I(\mathbf{r})$ and comparing it with a threshold value I_T. The lattice site at location $\mathbf{r}$ is open if $I(\mathbf{r}) \leq I_T$ and closed if $I(\mathbf{r}) > I_T$. To generate $I(\mathbf{r})$ they first generate a random function $I_0(\mathbf{r})$ with values at each lattice point drawn from identical, independent, uniform probability distributions. Then they take the convolution of I_0 with a smoothing function $K(\mathbf{r}|w)$ to obtain the correlated random function

$$I(\mathbf{r}) = \int K(\mathbf{r} - \mathbf{r}'|w) I_0(\mathbf{r}') d^d r', \tag{1}$$

where d is the dimension of the space. This procedure yields a correlated percolation lattice. Using the independent random function $I_0(\mathbf{r})$ rather than $I(\mathbf{r})$ gives ordinary percolation.

In Ref. [1] I reported simulations on cubic $N \times N \times N$ lattices with $N = 16$, 32, and 64, on square $N \times N$ lattices with $N = 256$, 512, and 1024, and on triangular and square 1-2 lattices with $N = 128$. (A square 1-2 lattice has both first- and second-neighbor connections.) Several different smoothing functions were used. In all cases, the percolation threshold p_c could be fit closely by a Gaussian function of the correlation length w

$$p_c = p_c^\infty + (p_c^0 - p_c^\infty) e^{-\alpha w^2}. \tag{2}$$

The parameters p_c^0 and p_c^∞ depend on the type of lattice but only slightly or not at all on the type of smoothing function. The parameter α depends on both the type of lattice and the type of smoothing function. The square and square 1-2 lattices are "matching lattices" as defined by Sykes and Essam [9,10] (p. 211) [11]. Sykes and Essam show that a lattice L and its matching lattice L^* satisfy the relation

$$p_c(L) + p_c(L^*) = 1. \tag{3}$$

If the percolation thresholds of matching lattices satisfy Eqs. (2) and (3), the limiting values p_c^0 and p_c^∞ also satisfy Eq. (3) and $\alpha(L^*) = \alpha(L)$. The results reported in Ref. [1] appear to agree with this, but, because of the small size of the square 1-2 lattice, one cannot draw a definite conclusion. For the triangular lattice, which is self-matching, Eq. (3) implies that $p_c = 1/2$ independent of w. This appears to be satisfied by the results of Ref. [1], but again the small lattice size precludes a definite conclusion.

To clarify the results for the square 1-2 and triangular lattices and to include all the standard two-dimensional lattices, I have performed simulations on large triangular, square 1-2, honeycomb, kagome, and brick wall lattices. Sketches of honeycomb, brick wall, and kagome lattices are

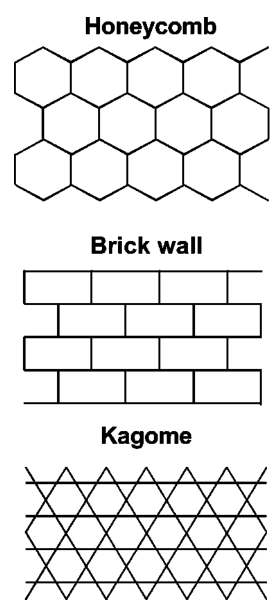

FIG. 1. Lattices.

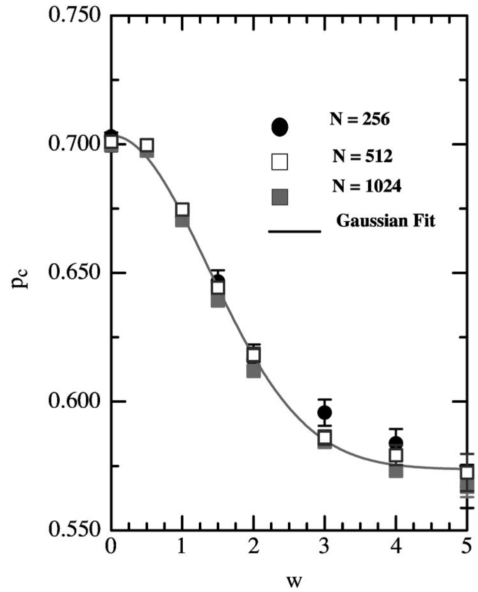

FIG. 2. Percolation thresholds for honeycomb lattices.

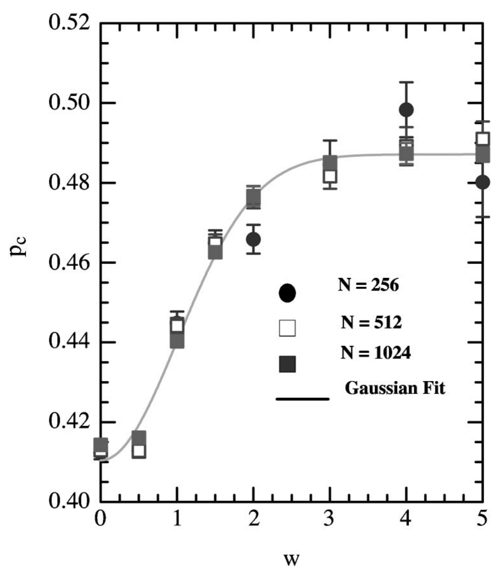

FIG. 3. Percolation thresholds for square 1-2 lattices.

TABLE I. Gaussian fit parameters.

Lattice	p_c^0	p_c^∞	α
Square [a,b]	0.595 ± 0.002	0.505 ± 0.002	0.408 ± 0.031
Square 1-2 [b]	0.410 ± 0.002	0.487 ± 0.002	0.49 ± 0.04
Honeycomb [b]	0.703 ± 0.002	0.574 ± 0.002	0.27 ± 0.01
Honeycomb [c]	0.698 ± 0.004	0.569 ± 0.003	0.48 ± 0.05
Brick wall [b]	0.704 ± 0.002	0.567 ± 0.002	0.35 ± 0.02
Kagome [b]	0.653 ± 0.003	0.559 ± 0.003	0.28 ± 0.03

[a]From Ref. [1].
[b]With the Gaussian smoothing function.
[c]With the Lorentzian smoothing function.

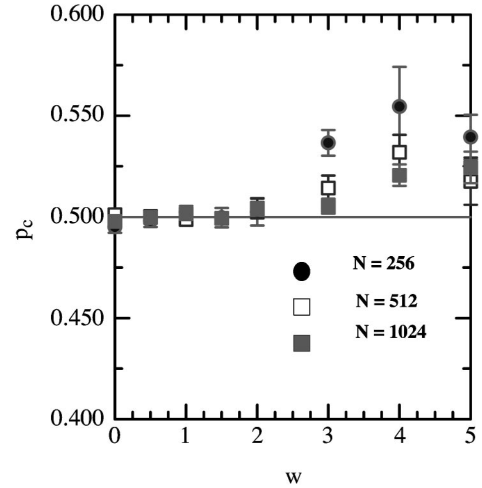

FIG. 4. Percolation thresholds for triangular lattices.

shown in Fig. 1. The honeycomb and brick wall lattices are topologically identical so they give the same results for independent lattice sites. However, the distances between sites are different in the two lattices so convolution leads to different results. Most simulations were preformed on $N \times N$ lattices with $N = 1024$. A few simulations with $N = 256$ or 512 gave essentially the same results. Since different smoothing functions give results that are qualitatively the same, most of the simulations were done only with the Gaussian smoothing function

$$K_G(\mathbf{r}|w) = e^{-r^2/w^2}. \tag{4}$$

For the honeycomb lattice, simulations were also performed using the Lorentzian smoothing function

$$K_L(\mathbf{r}|w) = (1 + r^2/w^2)^{-1}. \tag{5}$$

As in Ref. [1], the function $I_0(\mathbf{r})$ was obtained by calculating independent random numbers, uniformly distributed on the interval [0,1], for each lattice site using the FORTRAN library function RAN. The convolution of I_0 with the smoothing function was done using the fast Fourier transform [12]. And the percolation threshold was determined using the Hoshen-Kopelman algorithm [13–15].

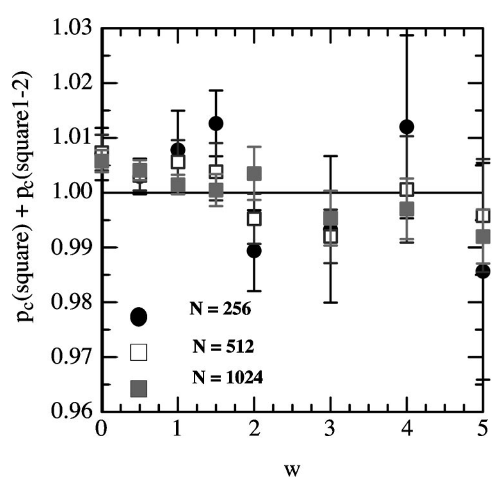

FIG. 5. Sum of percolation thresholds for matching lattices.

II. RESULTS AND DISCUSSION

Figures 2 and 3 show the dependence of percolation threshold on correlation length for, respectively, honeycomb and square 1-2 lattices, both with Gaussian smoothing functions. The solid line in each figure is a fit of the data to Eq. (2). Figures 2 and 3 are typical of all of the simulations on two-dimensional lattices. The percolation thresholds appear to fit on a single curve, independent of lattice size. The thresholds for $N=128$, which were left out to avoid cluttering the figures, also appear to fit on the same curve. There might be a weak dependence of threshold on the lattice size at large correlation lengths. However, the large uncertainties in the simulated percolation thresholds at larger correlation lengths and smaller lattice sizes make it impossible to draw a definite conclusion. In any case, one would expect the lattice size to become more significant at large correlation length. The only qualitative difference in the dependence of percolation threshold on correlation length is whether the percolation threshold increases or decreases with increasing correlation length. Equation (3) implies that if the percolation threshold of a lattice increases (decreases) with correlation length, the threshold of its matching lattice decreases (increases).

The simulated dependence of the percolation threshold on correlation length can be fit by Eq. (2). The parameters in this equation are listed in Table I. For comparison, I have also included the parameters for the square lattice from Ref. [1]. It appears that p_c^0 and p_c^∞ are dependent on the lattice type but not on the smoothing function; α is dependent on both lattice type and smoothing function.

For the triangular lattice, which is self-matching, Eq. (3) implies $p_c=1/2$ independent of correlation length. The simulated results, shown in Fig. 4, agree very well with this except for a small deviation at large correlation length which is probably an effect of lattice size. Taking an average of all the simulated data gives a value for the triangular lattice of $p_c=0.507\pm0.003$. Figure 5 shows the sum of the percolation thresholds for the square and square 1-2 lattices. The results agree very well with Eq. (3). Taking an average of all the data gives a value of $p_c(\text{square})+p_c(\text{square } 1-2)=0.998\pm0.002$. Comparing the Gaussian fit parameters in Table I we see that p_c^0 and p_c^∞ satisfy Eq. (3) as expected. However, contrary to expectations, $\alpha(\text{square})\neq\alpha(\text{square } 1-2)$. This suggests that the Gaussian function gives a phenomenological fit to the data, but does not have any theoretical significance.

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